

Introduction to mixed-effects models (for hierarchical data)

Nora Wickelmaier



2025-06-24

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- Try to go along in R! Ask as many questions as possible, also the ones you usually do not dare to ask (because you are supposed to know them already or something. . .)

Outline

① Introduction to random effects

② Hierarchical modeling

① Introduction to random effects

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 - Individuals are organized in groups (e.g., students nested in classes, or schools)
 - Persons are observed multiple times (observations nested in persons, longitudinal data)
- Statistical models for this kind of data are called multilevel models, mixed-effects models, random-effects models, covariance components models, or hierarchical models

Example: Mathematics achievement study

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- Hierarchical data structure
 - Students are organized in schools
 - y_{ij} mathematics achievement of student j in school i
 - x_{ij} (relative) socioeconomic status of student j in school i (overall mean 0, centered)

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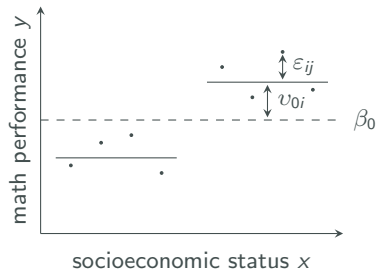
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 - Level 1: Student attributes
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Regression with random school effects

- What is the mean math achievement of the students?
- How much do schools vary in mean math achievement?

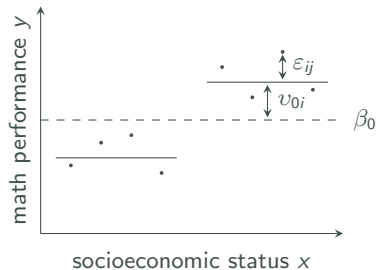
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Model equation

$$\text{(Level 1)} \quad y_{ij} = b_{0i} + \varepsilon_{ij}$$

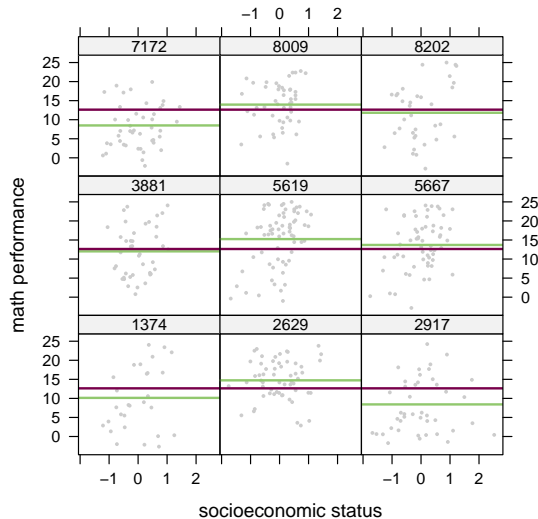
$$\text{(Level 2)} \quad b_{0i} = \beta_0 + v_{0i}$$

$$(2) \text{ in } (1) \quad y_{ij} = \beta_0 + v_{0i} + \varepsilon_{ij}$$

with $v_{0i} \sim N(0, \sigma_v^2)$ i.i.d, $\varepsilon_{ij} \sim N(0, \sigma^2)$ i.i.d,
 v_{0i} and ε_{ij} independent

Null model with random intercepts

Subset of 9 schools



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- School effects are therefore modeled as random effects (random variables) v_{0i}
- Only their variance σ_v^2 has to be estimated in the model
- The total variance of y_{ij} is decomposed into the variance between schools σ_v^2 and within schools σ^2

Results

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 - The estimated mean math achievement of students is $\hat{\beta}_0$
 - The estimated variance of schools in mean math achievement is $\hat{\sigma}_v^2$
 - The proportion of the total variance accounted for by the variance between schools is

$$\text{ICC} = \frac{\sigma_v^2}{\sigma_v^2 + \sigma^2}$$

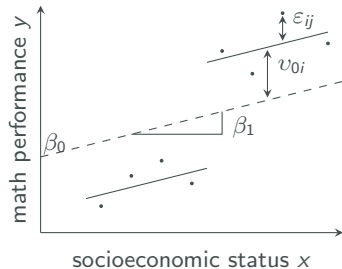
(Intra-class correlation)

Adding socioeconomic status as a predictor

- How strong is the relationship between students' socioeconomic status and their math achievement on average?
- How much do schools vary in mean math achievement for students with average socioeconomic status?

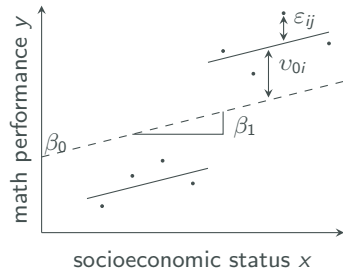
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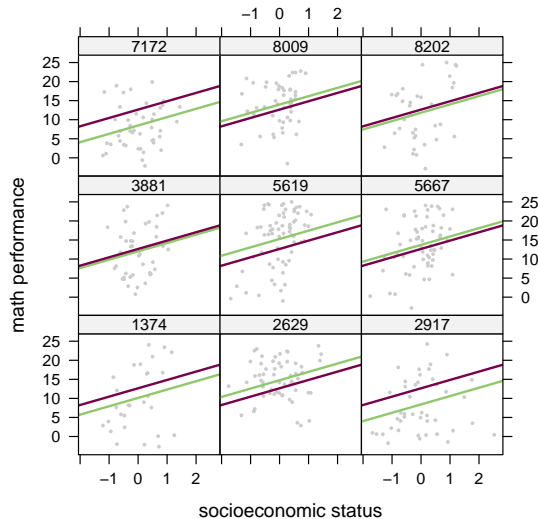
$$b_{1i} = \beta_1$$

$$\text{(2) in (1)} \quad y_{ij} = \beta_0 + \beta_1 x_{ij} + v_{0i} + \varepsilon_{ij}$$

with $v_{0i} \sim N(0, \sigma_v^2)$ i.i.d, $\varepsilon_{ij} \sim N(0, \sigma^2)$ i.i.d,
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Model with covariate and random intercepts

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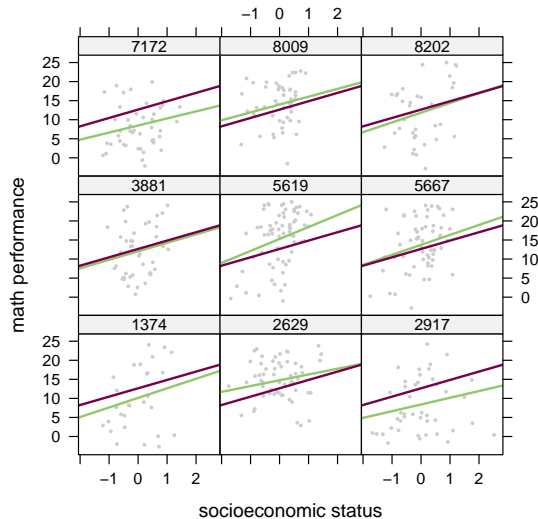
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- How do we add random slopes to a random intercept model using `lme4::lmer()`?
- Fit a model with random slopes for socioeconomic status in R

Model with covariate and random slopes

Subset of 9 schools



② Hierarchical modeling

HSB data set

- Two levels
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Level	Variable	Description
1	mathach	Performance in mathematics test
1	ses	(relative) socioeconomic status (overall mean 0)
2	meanses	mean socioeconomic status of the school (overall mean 0)
1	cses	Centered socioeconomic status of the student (mean for each school 0, difference $\text{ses} - \text{meanses}$)
2	school	school ID
2	sector	Public (0) or Catholic High School (1)

Hierarchical regression model

Model equation

$$\text{(Level 1)} \quad y_{ij} = b_{0i} + b_{1i} \text{cses}_{ij} + \varepsilon_{ij}$$

$$\text{(Level 2)} \quad b_{0i} = \beta_0 + \beta_2 \text{meanses}_i + \beta_4 \text{sector}_i + v_{0i}$$

$$b_{1i} = \beta_1 + \beta_3 \text{meanses}_i + \beta_5 \text{sector}_i + v_{1i}$$

$$\begin{aligned} \text{(2) in (1)} \quad y_{ij} = & \beta_0 + \beta_1 \text{cses}_{ij} + \beta_2 \text{meanses}_i + \beta_4 \text{sector}_i \\ & + \beta_3 (\text{cses}_{ij} \times \text{meanses}_i) + \beta_5 (\text{cses}_{ij} \times \text{sector}_i) \\ & + v_{0i} + v_{1i} \text{cses}_{ij} + \varepsilon_{ij} \end{aligned}$$

with

$$\begin{aligned} \begin{pmatrix} v_{0i} \\ v_{1i} \end{pmatrix} & \sim N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \mathbf{\Sigma}_v = \begin{pmatrix} \sigma_{v_0}^2 & \sigma_{v_0 v_1} \\ \sigma_{v_0 v_1} & \sigma_{v_1}^2 \end{pmatrix} \right) \text{ i.i.d} \\ \varepsilon_i & \sim N(\mathbf{0}, \sigma^2 \mathbf{I}_{n_i}) \text{ i.i.d} \end{aligned}$$

Decomposing socioeconomic status

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- Notice that the formulation of the model assumes dependencies of the slope b_{1i} on both mean socioeconomic status and school type, which is captured by the interactions of `cses` with `meanses` and `sector`, respectively

Exercise

1. Compute the model in R using `lme4::lmer()`

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2. Interpret the parameters

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Fixed effects

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 - The effect at the student level depends on the type of school: math achievement increases by 2.94 points in Public High Schools and by $2.94 - 1.64 = 1.30$ points in Catholic High Schools for a unit increase in `cses`
 - Higher math achievements are obtained in schools with higher mean socioeconomic status
 - In addition, the dependence of math achievement on `cses` scores is more pronounced in schools with higher `meanses` scores (estimated interaction > 0)

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- The corresponding covariance has an estimated value of $\hat{\sigma}_{v_0v_1} = 0.48 \cdot \hat{\sigma}_{v_0} \cdot \hat{\sigma}_{v_1} = 0.19$
- These results suggest a simplified model of the dependence of math achievement on cses, where the intercept, but not the slope varies across schools

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 - while individual differences are modeled, information aggregated over the sample is exploited, too
- Therefore, linear mixed-effects models allow for integrating differential and general psychological aspects within a common theoretical framework

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- You can send questions to me and even make an appointment with me to go over your solution

References

- Hedeker, D. R., & Gibbons, R. D. (2006). *Longitudinal data analysis*. John Wiley.
- Raudenbush, S. W., & Bryk, A. S. (2002). *Hierarchical linear models: Applications and data analysis methods* (Vol. 1). Sage.