Introduction to mixed-effects models

(for hierarchical data)

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→ Try to go along in R! Ask as many questions as possible, also the ones you usually do not dare to ask (because you are supposed to know them already or something. . .)

Outline

1. Introduction to random effects

2. Hierarchical modeling

1 Introduction to random effects

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  - Individuals are organized in groups (e.g., students nested in classes, or schools)
  - Persons are observed multiple times (observations nested in persons, longitudinal data)
- Statistical models for this kind of data are called multilevel models, mixed-effects models, random-effects models, covariance components models, or hierarchical models

Example: Mathematics achievement study

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- Hierarchical data structure
    - Students are organized in schools

      y_{ij} mathematics achievement of student j in school i

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  - Two levels
    - Level 1: Student attributes
    - Level 2: School attributes

Regression with random school effects

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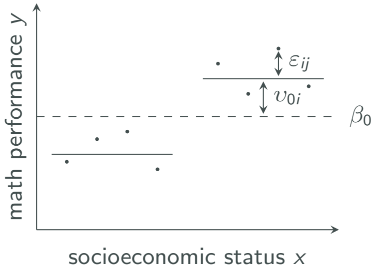

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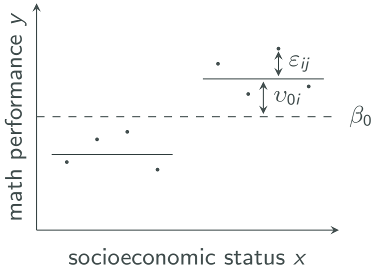

Model equation

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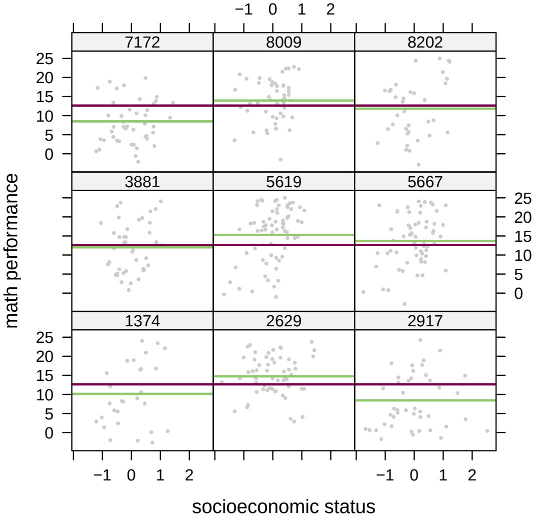
with $\upsilon_{0i} \sim N(0, \sigma_\upsilon^2)$ i.i.d, $\varepsilon_{ij} \sim N(0, \sigma^2)$ i.i.d, υ_{0i} and ε_{ij} independent

Null model with random intercepts

Subset of 9 schools



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- School effects are therefore modeled as random effects (random variables) υ_{0i}
- Only their variance σ_υ^2 has to be estimated in the model
- The total variance of y_{ij} is decomposed into the variance between schools σ_υ^2 and within schools σ^2

Results

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  - The estimated mean math achievement of students is $\hat{\beta}_0$
  - The estimated variance of schools in mean math achievement is $\hat{\sigma}_v^2$
  - The proportion of the total variance accounted for by the variance between schools is

$$\mathrm{ICC} = \frac{\sigma_v^2}{\sigma_v^2 + \sigma^2}$$

(Intra-class correlation)

Adding socioeconomic status as a predictor

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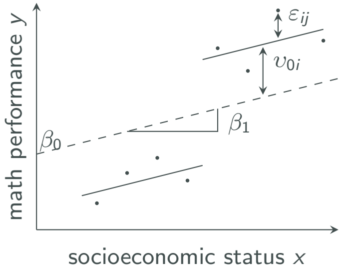

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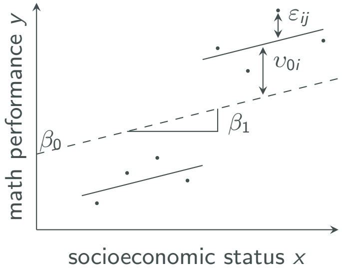

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Model with covariate and random intercepts

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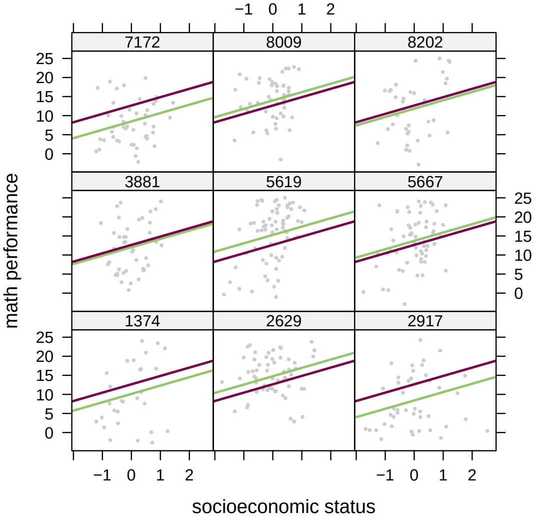

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- Fit a model with random slopes for socioeconomic status in R

Model with covariate and random slopes

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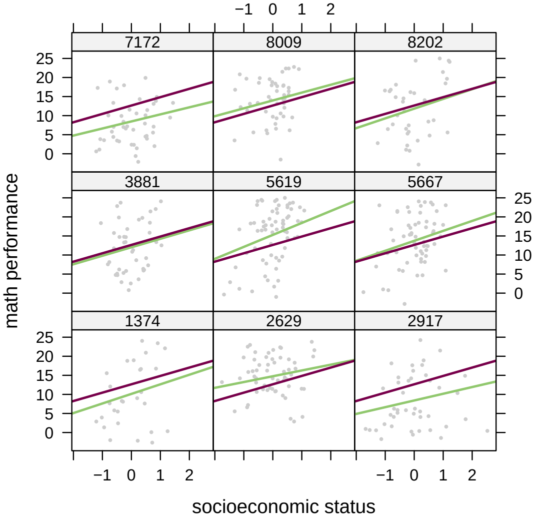

2 Hierarchical modeling

HSB data set

Level	Variable	Description
1	`mathach`	Performance in mathematics test
1	`ses`	(relative) socioeconomic status (overall mean 0)
2	`meanses`	mean socioeconomic status of the school (overall mean 0)
1	`cses`	Centered socioeconomic status of the student (mean for each school 0, difference `ses - meanses`)
2	`school`	school ID
2	`sector`	Public (0) or Catholic High School (1)

Hierarchical regression model

Model equation

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\text{(Level 1)} \quad y_{ij} &= b_{0i} + b_{1i}\, cses_{ij} + \varepsilon_{ij} \\
\text{(Level 2)} \quad b_{0i} &= \beta_0 + \beta_2 meanses_i + \beta_4 sector_i + \upsilon_{0i} \\
b_{1i} &= \beta_1 + \beta_3 meanses_i + \beta_5 sector_i + \upsilon_{1i} \\
\text{(2) in (1)} \quad y_{ij} &= \beta_0 + \beta_1\, cses_{ij} + \beta_2 meanses_i + \beta_4 sector_i \\
&\quad + \beta_3 (cses_{ij} \times meanses_i) + \beta_5 (cses_{ij} \times sector_i) \\
&\quad + \upsilon_{0i} + \upsilon_{1i} cses_{ij} + \varepsilon_{ij}
\end{aligned}
$$

with

$$
\begin{pmatrix} \upsilon_{0i} \\ \upsilon_{1i} \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \boldsymbol{\Sigma}_\upsilon = \begin{pmatrix} \sigma^2_{\upsilon_0} & \sigma_{\upsilon_0 \upsilon_1} \\ \sigma_{\upsilon_0 \upsilon_1} & \sigma^2_{\upsilon_1} \end{pmatrix} \right) \text{ i.i.d}
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Decomposing socioeconomic status

- In this model, by decomposing the socioeconomic status according to the equation

$$ses = cses + meanses$$

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- Notice that the formulation of the model assumes dependencies of the slope b_{1i} on both mean socioeconomic status and school type, which is captured by the interactions of `cses` with `meanses` and `sector`, respectively

Exercise

1. Compute the model in R using `lme4::lmer()`

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2. Interpret the parameters

Results

Fixed effects

- Mean math achievement (i.e., for a student with a mean `cses` score in a school with a mean `meanses` score) is 12.11 in Public High Schools and 13.33 in Catholic High Schools

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  - Higher math achievements are obtained in schools with higher mean socioeconomic status
  - In addition, the dependence of math achievement on `cses` scores is more pronounced in schools with higher `meanses` scores (estimated interaction > 0)

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- However, there is virtually no variation in the dependence of math achievement on `cses` across schools ($\hat{\sigma}^2_{\upsilon_1} = 0.07$), which should also be noted when interpreting the reported correlation of 0.48
- The corresponding covariance has an estimated value of $\hat{\sigma}_{\upsilon_0 \upsilon_1} = 0.48 \cdot \hat{\sigma}_{\upsilon_0} \cdot \hat{\sigma}_{\upsilon_1} = 0.19$
- These results suggest a simplified model of the dependence of math achievement on `cses`, where the intercept, but not the slope varies across schools

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    - while individual differences are modeled, information aggregated over the sample is exploited, too
- Therefore, linear mixed-effects models allow for integrating differential and general psychological aspects within a common theoretical framework

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   - You can send questions to me and even make an appointment with me to go over your solution

References

Hedeker, D. R., & Gibbons, R. D. (2006). *Longitudinal data analysis*. John Wiley.

Raudenbush, S. W., & Bryk, A. S. (2002). *Hierarchical linear models: Applications and data analysis methods* (Vol. 1). Sage.